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Interferencing in Coupled Bose–Einstein Condensates

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We consider an exactly soluble model of two Bose–Einstein condensates with a Josephson-type of coupling. Its equilibrium states are explicitly found showing condensation and spontaneously broken gauge symmetry. It is proved that the total number and total phase fluctuation operators, as well as the relative number and relative current fluctuation operators form both a quantum canonical pair. The exact relation between the relative current and phase fluctuation operators is established. Also the dynamics of these operators is solved showing the collapse and revival phenomenon.

KEY WORDS: Bose–Einstein condensation; fluctuations of a Josephson-type current; phase fluctuations; collapse and revivals; interferences in equilibrium.

1. INTRODUCTION

Since the 1995-observations[1–3] of Bose–Einstein condensation (BEC) in trapped alkali gases, an intense renewed interest is going on in the research of the physical properties and the nature of Bose condensed systems. In particular the interference pattern between two overlapping condensates has been discussed, see e.g., refs. 11–13, on the theoretical level.

In this context of interference, the static and dynamic properties of the phase of the condensate are of major importance. This has been the subject of many theoretical studies all over the last years. As a primordial and old question, the very existence of the phase and/or the phase operator, comes into the picture again.

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One encounters continuous efforts to formulate the phase (operator) in the standard theory of BEC, which we could call the Bogoliubov–Hartree–Fock theory,[5] in a system with a finite number of atoms (see e.g., ref. 6). One is constantly assuming that the condensation is occuring into a coherent state of the lowest energy mode of the system. Such a state fixes a well defined phase and amplitude, but should in stead exhibit inevitable fluctuations of both these quantities. Or, one is fixing the number of atoms in the system, i.e., the condensation takes place in a number state of Fock space, excluding any atom number fluctuations. Although these basic theoretical difficulties are now getting ripe in the minds of many researchers in the field, all kinds of procedures and tricks are permanently invented to wave away these difficulties. In this paper we take the point of view that these questions about the character of the quantum state into which the condensation occurs, and its major properties, are nevertheless of major importance. By now it is indeed well known that a condensate state is neither a "pure Fock," nor a "pure coherent" state in the strict mathematical sense, nor in the physical sense.

As explained above, "simple" coherent states or Fock states lead to annoying technical difficulties in order to describe and understand the essentials of many of the experimental challenging measurements on BEC which are constantly performed. After all, condensation is up to now, only clearly defined and generally accepted for homogeneous systems. Of course, we are aware of different tentatives to introduce decent thermodynamic limits for trapped gases. With all this knowledge in mind, we focus our attention here, not on the situation of BEC in trapped gases, but on the phenomenon of BEC for homogeneous systems, where one has a well defined thermodynamic limit, and where the occurence of BEC, accompanied by a spontaneous $U(1)$—symmetry breaking[7] is well understood.

Comparison of our point of view here with the recent studies based on the Gross–Pitaevskii equation (GPE) is at order. First of all we consider here temperature states, while the GPE is intended for ground state considerations. But even more basic, we consider homogeneous systems, while the GPE is an equation containing external fields and therefore yielding non-homogeneous equilibrium states. As explained above, the notion of BEC for homogeneous systems is clearly understood. To our knowledge, for non-homogeneous systems, rigorous proofs of BEC are still unavailable.

Furthermore we take into account that the main entries of the theory of the Bose condensates and their interference patterns are the *particle number fluctuations* and the *phase operator fluctuations*.

The main question is here, can one define rigorously a phase operator fluctuation and a particle number operator fluctuation of the condensate? The answer is proved to be positive. It is based on the notion of *fluctuation* *operator* which was introduced in a mathematically rigorous framework some time ago.[8, 9] We realize however that these results did not reach so far the majority of the theoretical physics community. The aim for introducing the notion of fluctuation operator, was precisely to study the quantum effects on the level of the fluctuations. We applied this theory of quantum fluctuation operators already in order to derive exactly rigorous results on BEC for the Bogoliubov–Hartree–Fock model.[10]

In Section 2, we describe these results in a language approachable for non-mathematics minded readers. The celebrated phase operator is nothing but the canonical fluctuation operator conjugate to the number operator fluctuation operator. This section is not just for warming up, but it should also shed a different light, than one is used to, on the status of the existence and meaning of the *phase operator*.

In Section 3, we study a model of two Bose–Einstein condensates with a Josephson-like coupling. We look at the static and thermodynamic properties of this model of two condensates. As far as we know, the problem of Josephson oscillations between coupled Bose–Einstein condensates has not yet been considered in a mathematically rigorous manner. Here we present a solvable model in a full quantum field theoretical setting and in Section 4 and 5 we give a full description and analysis of the dynamical equations of the total and relative number operator fluctuations and the total and relative phase operator fluctuations. About the dynamics we find the exact oscillatory time behavior of all these fluctuation operators. We also detect the so-called collapse and revival phenomena.

Our work is, as far as we know, the first rigorous one on this topic for homogeneous systems, it should also put in a new perspective much of the discussions which are going on in the large activity dealing with trapped Bosons and their interference patterns (see e.g., refs. 11–13). The main difference of these works with respect to ours, is that we consider fluctuations in the equilibrium state. We prove essentially a Nyquist-type of theorem, where in refs. 11–13 one considers the non-equilibrium problem of phase difference driven induced currents. In this case the dynamical equations are not closed, nor soluble, even not for the soluble model which we treat in the paper. On the other hand, by considering the fluctuations in the equilibrium situation, we proved that the dynamical equations for the order parameter fluctuation operator and the number fluctuation operator form a soluble closed dynamical system. Clearly, we present here a model of two soluble Bose systems interacting via a Josephson coupling. We conjecture that our result holds as well for coupled interacting Bose systems, i.e., we conjecture that the essentials of our result are model independent and omnipresent for coupled condensates. We shall come back on this at a later occasion.

2. NUMBER AND PHASE FLUCTUATION OPERATORS

In order to fix our ideas and in view of the model we describe in Section 3 for the study of the phase interference between two condensates, we present the number and phase fluctuation operators for the imperfect Bose gas (or mean field Bose gas).[16, 17] We follow the lines of ref. 10 but it should be clear that its validity is much larger.[14]

The leading idea of this section is to make clear that the up to now rather "mysterious" phase operator, which everybody uses in the field, but about which there are doubts on its very existence, has a firm mathematical definition in an equilibrium condensed state of a Bose gas. It should be realized that such a condensed state is neither a coherent, nor a Fock state in the technical narrow sense. It is defined as the *fluctuation operator canonically adjoint to the number fluctuation operator*. This definition is not just a formal thing, but it gives a physical interpretation of the phase operator, different from the existing ones. We are not repeating here all the mathematics of the definition of the phase fluctuation operator, which can be found in various papers (see e.g., refs. 9 and 15). We content ourself here in making these definitions plausible for the imperfect Bose gas.

Let $\Lambda \subset \mathbb{R}^3$ be the centered cubic box of length L, with periodic boundary conditions. The Boson creation and annihilation operators in the one-particle state $\psi_{L,k}(x) = V^{-1/2} e^{ik.x}$, $x \in \Lambda$, $k \in \Lambda^* = (2\pi/L)\, \mathbb{Z}^\nu$ are given by

$$a^*(\psi_{L,k}) \equiv a^*_{L,k} = \frac{1}{\sqrt{V}} \int_\Lambda dx\, a^*(x)\, e^{ik.x}$$

and

$$a(\psi_{L,k}) \equiv a_{L,k} = \frac{1}{\sqrt{V}} \int_\Lambda dx\, a(x)\, e^{-ik.x}$$

with

$$[a(x), a^*(y)] = \delta(x-y)$$

The imperfect Bose gas is specified by the local Hamiltonian H_L:[18]

$$H_L = T_L - \mu_L N_L + \frac{\lambda}{2V} N_L^2 \tag{1}$$

where

$$T_L = \sum_{k \in \Lambda^*} \varepsilon_k a^*_{L,k} a_{L,k}, \qquad \varepsilon_k = \frac{|k|^2}{2m}$$

$$N_L = \sum_{k \in \Lambda^*} a^*_{L,k} a_{L,k}$$

$\lambda > 0$ measures the strength of the mean field inter particle repulsion.

This model is exactly soluble in the thermodynamic limit $L \to \infty$, keeping the particle density in the Gibbs state $\omega_L(\cdot)$ for (1) constant, equal to ρ, i.e., for all L:

$$\frac{1}{V} \omega_L(N_L) = \rho$$

It is proved rigorously[18] that for $T < T_c$ or ρ large enough and $\nu \geqslant 3$, the limit state $\omega_\beta(\cdot) = \lim_{L \to \infty} \omega_L(\cdot)$ exists as an integral over ergodic states $(\omega^\alpha_\beta,\ \alpha \in [0, 2\pi])$:

$$\omega_\beta(\cdot) = \frac{1}{2\pi} \int_0^{2\pi} d\alpha\, \omega^\alpha_\beta(\cdot)$$

with

$$\omega^\alpha_\beta(e^{i[a^*(f) + a(f)]}) = e^{-(1/2)(f, Kf) + 2i\sqrt{\rho_0}\, |\hat{f}(0)| \cos \alpha}$$

where

$$\widehat{Kf}(k) = \frac{1}{2} \hat{f}(k) \coth \frac{\beta \varepsilon_k}{2}, \qquad f \in L^2(\mathbb{R}^\nu)$$

The spontaneous gauge symmetry breaking accompanying the phase transition is visible in the states ω^α_β having the property:

$$\lim_{L \to \infty} \omega^\alpha_\beta \left(\frac{a^*_{L,0}}{\sqrt{V}} \right) = \sqrt{\rho_0}\, e^{i\alpha}$$

Clearly, ρ_0 is the condensate and α is the phase of the order parameter. One also proves that in the state ω^α_β, one has the operator limit:

$$\lim_{L \to \infty} \frac{a^*_{L,0}}{\sqrt{V}} = \sqrt{\rho_0}\, e^{i\alpha}$$

From now on we limit our attention to one of the ergodic states ω_β^α for some fixed α, and without restriction of generality we take $\alpha=0$, and with a condensate density $\rho_0 \neq 0$. For simplicity, denote the state ω_β^0 by ω.

The state ω does not have the gauge symmetry. The generator of the gauge symmetry is the number operator

$$N_L = \int_\Lambda dx\, a^*(x)\, a(x)$$

with local number density operator $n(x) = a^*(x)\, a(x)$. The common choice of order parameter operator is $V^{-1/2} a^\sharp_{L,0}$, or taking a self-adjoint combination:

$$O_L = \frac{i}{\sqrt{V}} (a^*_{L,0} - a_{L,0}) = \frac{i}{V} \int_\Lambda dx (a^*(x) - a(x))$$

with local order parameter density operator $o(x) = i(a^*(x) - a(x))$.

We concentrate now on the k-mode fluctuations, with $k \neq 0$, of the local number and order parameter density operators, i.e., on

$$F_{L,k}(n) = \frac{1}{\sqrt{V}} \int_\Lambda dx (n(x) - \omega(n(x))) \cos k.x$$

$$F_{L,k}(o) = \frac{1}{\sqrt{V}} \int_\Lambda dx (o(x) - \omega(o(x))) \cos k.x$$

Remark that for all finite L, the quantities $F_{L,k}(n)$ and $F_{L,k}(o)$ are operators and do represent the fluctuations of the number density and of the order parameter density.

The first tedious question that is posed, is to characterize the limit operators:

$$F_k(n) = \lim_{L\to\infty} F_{L,k}(n)$$

$$F_k(o) = \lim_{L\to\infty} F_{L,k}(o)$$

The details of the proof of these limits can be found in ref. 10. Here we just mention that the limits are taken in the sense of a central limit theorem. The main result is that the limits $F_k(n)$ and $F_k(o)$ are operators on a well specified Hilbert space, $\tilde{\mathscr{H}}_k$, generated by a normalized vector $\tilde{\Omega}_k$ and vectors

$F_k(A_1)\cdots F_k(A_n)\,\tilde{\Omega}_k$, with the A_i local operators, like e.g., $n(x)$ and $o(x)$, and with arbitrary n; the scalar product of $\mathscr{H}_k$ is given by

$$(F_k(A_1)\cdots F_k(A_n)\,\tilde{\Omega}_k, F_k(B_1)\cdots F_k(B_n)\,\tilde{\Omega}_k) = \delta_{n,m}\,\mathrm{Perm}((\tilde{\Omega}_k, F_k(A_i)\,F_k(B_j)\,\tilde{\Omega}_k))_{i,j} \tag{2}$$

with two-point function given by

$$(\tilde{\Omega}_k, F_k(A_i)\,F_k(B_j)\,\tilde{\Omega}_k) = \lim_{L\to\infty} \omega(F_{L,k}(A_i)\,F_{L,k}(B_j)) \tag{3}$$

i.e., essentially determined by the two-point functions of the given state ω. Therefore as is clear from (2), all $(n+m)$-point functions are given by the two-point function (3). The definition (2) defines completely the fluctuation operators $F_k(A)$ on the Hilbert space $\tilde{\mathscr{H}}_k$.

On the other hand, the non-commutative law of large numbers, here of large operators, leads straightforwardly to the canonical commutation relation

$$\lim_{L\to\infty} [F_{L,k}(n), F_{L,k}(o)] = \lim_{L\to\infty} \frac{1}{2V} \int_\Lambda dx[n(x), o(x)] = \omega([n(x), o(x)]) = i\sqrt{\rho_0}$$

or

$$[F_k(n), F_k(o)] = i\sqrt{\rho_0} \tag{4}$$

This is the basic result for the definition of the phase operator. Equation (4) means that the number fluctuation operator $F_k(n)$ and the order parameter fluctuation operator $F_k(o)$ are canonically conjugate (compare with $[q, p] = i\hbar$). Hence for the physics of BEC we found on the level of fluctuations the canonical pair $(F_k(n), F_k(o))$. Clearly the operator $F_k(o)$ satisfies all basic physical requirements for playing the role of what is usually called the *phase operator* of the condensate.

The reader will have recognized from (2) and (4) that the fluctuation operators

$$F_k(A), F_k(B), \ldots$$

form an algebra of Boson field operators and in particular that $F_k(n)$ and $F_k(o)$ form quantum canonical variables with a quantization parameter

$\sqrt{\rho_0}$ (compare with $\hbar$). On the other hand, from (2) it is clear that the vector $\tilde{\Omega}_k$ defines a generalized or quasi free (Gaussian) state

$$\tilde{\omega}_k(\cdot) = (\tilde{\Omega}_k, \cdot \tilde{\Omega}_k)$$

on the Boson field algebra (see ref. 19).

This means that the central limit theorem for the k-mode fluctuations in the state ω defines an equilibrium state $\tilde{\omega}_k$ on the fluctuation operators. The mentioned quasi free character means that all correlation functions of limit fluctuation operators are polynomial functions only of the one- and two-point functions. For more details we refer once more to ref. 10.

The reader might ask for the unicity of this *phase operator*, if being defined only as the canonically adjoint operator to the number fluctuation operator. The interested reader is referred to Section 5 and refs. 10 and 14 for that discussion.

3. THE MODEL AND EQUILIBRIUM STATES

We consider two coupled Bose–Einstein condensates, each of them modeled by an imperfect or mean field Bose gas. Denote $a_i^\#(x)$, $i = 1, 2$ the creation and annihilation operators for the two Bose gases, i.e.,

$$[a_i(x), a_j^*(y)] = \delta_{i,j}\,\delta(x-y), \qquad x, y \in \mathbb{R}^3$$

We assume that the two gases have the same particle density $\rho/2$ (hence ρ is the total particle density $\rho = (1/V)\,\omega_L(N_L)$ with $N_L = N_{1,L} + N_{2,L}$), and also that they are of the same type of particles (i.e., there is only one mean field constant λ). We also assume a phase difference φ between the gases, and model the Josephson coupling between the gases by a term

$$C_L^{1,2} = -\gamma \sum_{k \in \Lambda^*} a_{1,k}^* a_{2,k} e^{-i\varphi} + a_{2,k}^* a_{1,k} e^{i\varphi} \tag{5}$$

with $\gamma > 0$ the coupling constant.

Hence the local Hamiltonian of the system we study is given by:

$$\begin{aligned} H_L &= T_{1,L} + T_{2,L} - \mu_L N_L + \frac{\lambda}{2V} N_L^2 + C_L^{1,2} \\ &= \sum_{k \in \Lambda^*} (\varepsilon_k - \mu_L)(a_{1,k}^* a_{1,k} + a_{2,k}^* a_{2,k}) + \frac{\lambda}{2V}(N_{1,L} + N_{2,L})^2 \\ &\quad - \gamma \sum_{k \in \Lambda^*} a_{1,k}^* a_{2,k} e^{-i\varphi} + a_{2,k}^* a_{1,k} e^{i\varphi} \end{aligned} \tag{6}$$

In this section we find the limiting Gibbs states $\omega = \lim_{L\to\infty} \omega_L$ at inverse temperature β of this model. A rigorous study along the lines of ref. 18 is perfectly possible, but we permit ourselves here a more intuitive approach. As in any mean field model, we replace the Hamiltonian (6) by a state dependent effective Hamiltonian

$$H_L^\omega = \sum_{k\in\Lambda^*} (\varepsilon_k - \mu + \lambda\rho)(a_{1,k}^* a_{1,k} + a_{2,k}^* a_{2,k})$$

$$-\gamma \sum_{k\in\Lambda^*} a_{1,k}^* a_{2,k} e^{-i\varphi} + a_{2,k}^* a_{1,k} e^{i\varphi} \tag{7}$$

with $\mu = \lim_{L\to\infty} \mu_L$ in correspondence with the constraint $\rho = (1/V)\,\omega(N_L)$. This effective Hamiltonian is bilinear in the creation and annihilation operators and therefore it can be diagonalized by a Bogoliubov transformation.

Let $\delta_L^\omega(\cdot) = [H_L^\omega, \cdot\,]$ be the generator of this dynamics and $f_k = \varepsilon_k - \mu + \lambda\rho$. Then

$$\delta_L^\omega \begin{pmatrix} a_{1,k}^* \\ a_{2,k}^* \end{pmatrix} = \begin{pmatrix} f_k & -\gamma e^{i\varphi} \\ -\gamma e^{-i\varphi} & f_k \end{pmatrix} \begin{pmatrix} a_{1,k}^* \\ a_{2,k}^* \end{pmatrix}$$

The matrix

$$\begin{pmatrix} f_k & -\gamma e^{i\varphi} \\ -\gamma e^{-i\varphi} & f_k \end{pmatrix}$$

has eigenvalues $E_k^\pm$,

$$E_k^\pm = f_k \pm \gamma$$

with corresponding eigenoperators $b_{\pm,k}^*$,

$$b_{\pm,k}^* = \frac{1}{\sqrt{2}} (a_{1,k}^* e^{-(i/2)\varphi} \mp a_{2,k}^* e^{(i/2)\varphi}) \tag{8}$$

i.e.,

$$\delta_L^\omega(b_{\pm,k}^*) = E_k^\pm\, b_{\pm,k}^* \tag{9}$$

The $b_\pm^\sharp(x)$ still satisfy Boson commutation rules. The energy spectrum of the quasi-particles $b_\pm^\sharp(x)$ has two branches, $\{E_k^\pm \mid k \in \mathbb{R}^\nu\}$.

By a standard argument, using the Boson commutation rules and the correlation inequalities,[20, 21] characterizing the limit equilibrium states,

$$\lim_{L\to\infty} \beta\omega(X^*\delta_L^{\omega}(X)) \geqslant \omega(X^*X) \ln \frac{\omega(X^*X)}{\omega(XX^*)} \tag{10}$$

for X any polynomial in the creation and annihilation operators, one finds in a straightforward manner:

$$\omega(b^*_{\pm,k} b_{\pm,k}) = \frac{1}{e^{\beta E_k^{\pm}} - 1} \tag{11}$$

Along the usual lines of the derivation of Bose–Einstein condensation, we find a critical density (or inverse temperature) above which one derives the following value of the chemical potential:

$$\mu = \lambda\rho - \gamma$$

i.e., $E_k^- = \varepsilon_k$ and $E_k^+ > 2\gamma$ for all k. Hence there is a macroscopic occupation of the 0-momentum state of the "$-$"-mode possible; the condensate density is given by:

$$\lim_{V\to\infty} \frac{1}{V} \omega(b^*_{-,0} b_{-,0}) = \rho_0 > 0 \tag{12}$$

There is no condensation for the "$+$"-mode since $\lim_{k\to 0} E_k^+ = 2\gamma > 0$ for $\mu = \lambda\rho - \gamma$.

From $\rho = \lim_{L\to\infty} V^{-1}\omega(N_L)$ and (8), one finds

$$\begin{aligned} \rho &= \lim_{L\to\infty} \frac{1}{V} \sum_k \omega(b^*_{-,k} b_{-,k}) + \omega(b^*_{+,k} b_{+,k}) \\ &= \rho_0 + \int_{\mathbb{R}^\nu} \frac{dk}{(2\pi)^3} \frac{1}{e^{\beta\varepsilon_k} - 1} + \int_{\mathbb{R}^\nu} \frac{dk}{(2\pi)^3} \frac{1}{e^{\beta(\varepsilon_k + 2\gamma)} - 1} \end{aligned}$$

For $\alpha \geqslant 0$, denote

$$\rho(\alpha) = \int_{\mathbb{R}^\nu} \frac{dk}{(2\pi)^3} \frac{1}{e^{\beta(\varepsilon_k + \alpha)} - 1} \tag{13}$$

then (12) becomes

$$\rho_0 = \rho - \rho(0) - \rho(2\gamma)$$

It is clear that $\rho(0) + \rho(2\gamma)$ is the critical density.

The Bose–Einstein condensation (12) implies a spontaneous breaking of the gauge symmetry,[7] i.e., the limiting Gibbs state ω decomposes with respect to the $U(1)$ gauge group into distinct extremal equilibrium states:

$$\omega = \frac{1}{2\pi} \int_0^{2\pi} d\theta \, \omega_\theta$$

where each of the states ω_θ is determined by the two-point function (11) and the one-point function

$$\omega_\theta(b_-^*(x)) = \sqrt{\rho_0} \, e^{i\theta} \tag{14}$$

This important point, whose non-triviality is often overlooked, is proved in ref. 7. Of course, the other way round, namely that gauge symmetry breaking implies condensation, is well known and follows trivially from the Schwartz inequality.

From now on we choose a particular extremal equilibrium state ω_θ, and without loss of generality we take $\theta = 0$. For notational simplicity, this state is again denoted by ω. Using once more the correlation inequalities (10), it is not difficult to show that the higher order correlations decompose into sums and products of one-and two-point correlations, given by (14) respectively (11), i.e., the state ω is quasi-free. Therefore we have completely characterized the equilibrium states.

It is clear that the gauge symmetry breaking state under discussion indeed corresponds to a state of two different condensates interacting through a Josephson coupling (5). Since $\omega(b_+^*(x)) = 0$ and $\omega(b_-^*(x)) = \sqrt{\rho_0}$, we find using (8)

$$\begin{aligned} \omega(a_1^*(x)) \, e^{-(i/2)\,\varphi} &= \omega(a_2^*(x)) \, e^{(i/2)\,\varphi} \\ &= \sqrt{\frac{\rho_0}{2}} \end{aligned} \tag{15}$$

Notice that φ is indeed the phase difference between the condensates. Our arbitrary choice $\theta = 0$ then actually determines both phases to be $\pm \varphi/2$ and the choice of equal particle densities $\rho/2$ yields equal condensate densities $\rho_0/2$. Moreover it is obvious that the gap-less mode E_k^- is related to the broken gauge symmetry, i.e., to the Bose–Einstein condensation, and that the mode E_k^+ with energy gap 2γ arises due to the presence of the Josephson coupling. The detailed study of the fluctuation operators corresponding to these two excitation branches and modes is the subject of the subsequent sections.

4. TOTAL NUMBER AND PHASE FLUCTUATION OPERATORS

Motivated by the discussion in Section 2 and ref. 10, define the total number and phase fluctuations in the box Λ by $(k \neq 0)$

$$F_{L,k}(n_{\text{tot}}) = \frac{1}{\sqrt{V}} \int_\Lambda dx (a_1^*(x)\, a_1(x) + a_2^*(x)\, a_2(x) - \rho) \cos k.x \tag{16}$$

$$F_{L,k}(\phi_{\text{tot}}) = \frac{i}{\sqrt{V}} \int_\Lambda dx (b_-^*(x) - b_-(x)) \cos k.x \tag{17}$$

where $b_-(x)$ is defined in (8).

Again on the basis of the law of large numbers:

$$\lim_{L\to\infty} [F_{L,k}(n_{\text{tot}}), F_{L,k}(\phi_{\text{tot}})] = \lim_{L\to\infty} \frac{i}{2\sqrt{V}} (b_{-,0}^* + b_{-,0}) = i\sqrt{\rho_0} \tag{18}$$

Using (8) one writes also

$$a_1^*(x)\, a_1(x) + a_2^*(x)\, a_2(x) = b_-^*(x)\, b_-(x) + b_+^*(x)\, b_+(x)$$

or

$$F_{L,k}(n_{\text{tot}}) = F_{L,k}(n_-) + F_{L,k}(n_+)$$

with $n_\pm$ defined in the obvious sense. Hence the total number operator fluctuation is the sum of two number operator fluctuations of two imperfect Bose gases. For a single imperfect Bose gas (see Section 2), one finds this analysis as the subject of ref. 10, making the present analysis straightforward.

As already discussed in Section 2, and apparent from (18), the limiting fluctuation operators $F_k(\cdot)$ satisfy Bosonic commutation rules (although the *local* fluctuation operators *do not*). Equation (18) learns also that the fluctuation operators $F_k(n_{\text{tot}})$ and $F_k(\phi_{\text{tot}})$ constitute a canonical pair, generating an algebra of canonical commutation relations (CCR) of fluctuation observables of the system. Furthermore, it is shown in refs. 8 and 9 and briefly discussed in Section 2 that the central limit theorem also fixes an equilibrium state $\tilde{\omega}_k$ on this algebra of limiting fluctuation operators, which is a CCR-algebra of Bosonic field operators. This state is shown to be quasi-free and gauge invariant, and hence completely determined by its two-point function, given by:[8, 9]

$$\tilde{\omega}_k(F_k(A)\, F_k(B)) = \lim_{L\to\infty} \int_\Lambda dz\, \omega(A(z)\, B(0)) \cos k.z$$

where A, B are (in the present case) polynomials in the microscopic canonical Bosonic field operators. Remark that there are no technical problems related to the central limit theorem for $k\neq 0$, off-diagonal long-range order correlations do appear only at $k=0$.

The first step in our study of the total number- and phase fluctuation operators is to determine their variances.

Proposition 1. For $k\neq 0$ we have

(i) $\tilde{\omega}_k(F_k(n_{\text{tot}})^2)=\tilde{\omega}_k(F_k(n_-)^2)+\tilde{\omega}_k(F_k(n_+)^2)$

$$=\frac{\rho_0}{2}\coth\frac{\beta\varepsilon_k}{2}+\frac{1}{2}\int_{\mathbb{R}^\nu}\frac{dp}{(2\pi)^3}\frac{e^{\beta\varepsilon_{p+k}}+e^{\beta\varepsilon_p}}{(e^{\beta\varepsilon_{p+k}}-1)(e^{\beta\varepsilon_p}-1)}$$

$$+\frac{1}{2}\int_{\mathbb{R}^\nu}\frac{dp}{(2\pi)^3}\frac{e^{\beta E^+_{p+k}}+e^{\beta E^+_p}}{(e^{\beta E^+_{p+k}}-1)(e^{\beta E^+_p}-1)} \quad (19)$$

(ii) $$\tilde{\omega}_k(F_k(\phi_{\text{tot}})^2)=\frac{1}{2}\coth\frac{\beta\varepsilon_k}{2} \quad (20)$$

Proof. (i) The fact that $\tilde{\omega}_k(F_k(n_{\text{tot}})^2)=\tilde{\omega}_k(F_k(n_-)^2)+\tilde{\omega}_k(F_k(n_+)^2)$ follows from the fact that the "$+$"- and "$-$"-mode are independent of each other. For a calculation of the explicit expression for $\tilde{\omega}_k(F_k(n_{\text{tot}})^2)$, see ref. 10.

(ii) This is simply the two-point function of the state ω, see ref. 10. ▮

Remark that the result of this proposition, supplemented with (18) is sufficient in order to characterize completely the limiting fluctuation operators $F_k(n_{\text{tot}})$ and $F_k(\phi_{\text{tot}})$ on a well defined Hilbert space (see Section 2). We do not enter into these technical details.

Of course we are interested particularly in the limit $k\to 0$ of these operators, in order to see how the long-range order due to the Bose–Einstein condensation manifests itself on the level of the fluctuations. In ref. 10 one can find a discussion demonstrating that quantum effects will only be present in the limit $k\to 0$ if one works in the ground state ω^g, defined as the zero-temperature limit of the equilibrium state ω:

$$\omega^g=\lim_{\beta\to\infty}\omega$$

At $T=0$, the variances (19) and (20) simplify to

$$\tilde{\omega}_k^g(F_k(n_{\rm tot})^2) = \frac{\rho_0}{2}$$

$$\tilde{\omega}_k^g(F_k(\phi_{\rm tot})^2) = \frac{1}{2}$$

and the limit $k \to 0$ is trivial:

$$F_0(n_{\rm tot}) = \lim_{k \to 0} F_k(n_{\rm tot}) \qquad \text{and} \qquad F_0(\phi_{\rm tot}) = \lim_{k \to 0} F_k(\phi_{\rm tot})$$

These are well defined fluctuation operators, satisfying

$$[F_0(n_{\rm tot}), F_0(\phi_{\rm tot})] = i\sqrt{\rho_0}$$

$$\tilde{\omega}_0^g(F_0(n_{\rm tot})^2) = \frac{\rho_0}{2}$$

$$\tilde{\omega}_0^g(F_0(\phi_{\rm tot})^2) = \frac{1}{2}$$

We derived the exact uncertainty relation between the number operator and phase operator, given by

$$\tilde{\omega}_0^g(F_0(n_{\rm tot})^2)\,\tilde{\omega}_0^g(F_0(\phi_{\rm tot})^2) = \frac{\rho_0}{4}$$

Remark that the condensate density ρ_0, in fact $\sqrt{\rho_0}$, acts in this equation, as well as in Eq. (18), as a quantization parameter (compare with $\hbar$). Consequently, the whole content of our results, as well as all physical interpretations, do disappear in the absence of condensation, i.e., if $\rho_0 = 0$.

Finally remark that we have omitted in our notation the state dependence of the fluctuation operators throughout, although this dependence is important. Fluctuation operators corresponding to different states (e.g., corresponding to different temperature or different phase) in fact are not comparable as they act on completely different Hilbert spaces. We do not enter into these mathematical subtleties.

5. RELATIVE NUMBER AND PHASE FLUCTUATION OPERATORS

The relative number operator in a finite volume is:

$$N_{L,\,\rm rel} = N_{1,\,L} - N_{2,\,L}$$

and its k-mode fluctuation

$$F_{L,k}(n_{\mathrm{rel}}) = \frac{1}{V^{1/2}} \int_{\Lambda} dx (a_1^*(x)\, a_1(x) - a_2^*(x)\, a_2(x)) \cos k.x$$

As before, we are primarily interested in the limit $L \to \infty$, followed by the limit $k \to 0$. The relative number operator $N_{L,\mathrm{rel}}$ is not the generator of a symmetry of the local Hamiltonian H_L (6) because of the Josephson coupling term. This means that there is no question of spontaneous symmetry breaking for the relative number operator, as it is not a symmetry. Furthermore a straightforward computation of the dynamics of $N_{L,\mathrm{rel}}$ learns that its spectrum belongs to the excitation branch $E^+ - E^-$. Since $E_k^+ - E_k^- = 2\gamma > 0$, the spectrum of $N_{L,\mathrm{rel}}$ shows an energy gap. Hence the 0-mode fluctuations of the relative number operator and its adjoint will be normal. Therefore the limits $L \to \infty$ and $k \to 0$ may be interchanged and hence the starting point of our investigation can be the operator (i.e., the case $k = 0$):

$$F_L(n_{\mathrm{rel}}) = \frac{1}{V^{1/2}} \int_{\Lambda} dx (a_1^*(x)\, a_1(x) - a_2^*(x)\, a_2(x))$$

$$= \frac{1}{V^{1/2}} \sum_k a_{1,k}^* a_{1,k} - a_{2,k}^* a_{2,k} \tag{21}$$

Next define the fluctuation operator of the relative current:

$$F_L(j_{\mathrm{rel}}) = i[H_L^{\varphi}, F_L(n_{\mathrm{rel}})]$$

$$= \frac{2i\gamma}{V^{1/2}} \int_{\Lambda} dx (a_1^*(x)\, a_2(x)\, e^{-i\varphi} - a_2^*(x)\, a_1(x)\, e^{i\varphi})$$

$$= \frac{2i\gamma}{V^{1/2}} \sum_k a_{1,k}^* a_{2,k} e^{-i\varphi} - a_{2,k}^* a_{1,k} e^{i\varphi} \tag{22}$$

This operator clearly corresponds to the 0-mode fluctuations of the relative current from one gas into the other. Below, we show in a series of steps that the operators (21) and (22) are each others adjoint in the limit $L \to \infty$. Afterwards we show the relations between the relative current fluctuation operator on the one hand and the relative phase fluctuation operator on the other hand.

A central limit theorem and reconstruction theorem can again be proved for these operators (see refs. 9 and 10), proving the existence of the Bosonic field operators

$$F(n_{\mathrm{rel}}) = \lim_{L \to \infty} F_L(n_{\mathrm{rel}})$$

$$F(j_{\mathrm{rel}}) = \lim_{L \to \infty} F_L(j_{\mathrm{rel}})$$

in a rigorous mathematical sense.

Let $\delta_\omega(\cdot) = \lim_{L \to \infty} [H_L^\omega, \cdot\,]$ be the infinitesimal generator of the dynamics (H_L^ω is the effective Hamiltonian defined in Eq. (7), with $\mu = \lambda\rho - \gamma$).

In order to prove the properties below, it is convenient to write (21) and (22) in terms of the quasi-particle operators (8):

$$F_L(n_{\mathrm{rel}}) = \frac{1}{V^{1/2}} \sum_k b^*_{+,k} b_{-,k} + b^*_{-,k} b_{+,k} \tag{23}$$

$$F_L(j_{\mathrm{rel}}) = \frac{2i\gamma}{V^{1/2}} \sum_k b^*_{+,k} b_{-,k} - b^*_{-,k} b_{+,k} \tag{24}$$

Proposition 2. The operators $F(n_{\mathrm{rel}})$ and $F(j_{\mathrm{rel}})$ form a canonical pair and satisfy

$$[F(n_{\mathrm{rel}}), F(j_{\mathrm{rel}})] = ic_{\mathrm{rel}}$$

where

$$c_{\mathrm{rel}} = 4\gamma(\rho_0 + \rho(0) - \rho(2\gamma)) > 0$$

and $\rho(\alpha)$ is defined in (13).

Proof. This follows again from the general theory on normal fluctuation operators.(9, 10) The explicit expression for c_{rel} follows from

$$\omega([F_L(n_{\mathrm{rel}}), F_L(j_{\mathrm{rel}})]) = \frac{4i\gamma}{V} \sum_k \omega(b^*_{-,k} b_{-,k} - b^*_{+,k} b_{+,k})$$

$$\xrightarrow{L \to \infty} 4i\gamma(\rho_0 + \rho(0) - \rho(2\gamma)) \quad \blacksquare$$

The infinitesimal generator δ_ω of the micro-dynamics induces a natural infinitesimal generator $\tilde{\delta}_\omega$ of a dynamics on the macroscopic fluctuation operators by the formula:[9]

$$\tilde{\delta}_\omega F(A) = F(\delta_\omega(A))$$

Proposition 3. The infinitesimal generator $\tilde{\delta}_\omega$ on the macroscopic fluctuations is given by:

$$\tilde{\delta}_\omega F(n_{\text{rel}}) = -iF(j_{\text{rel}}) \tag{25}$$

$$\tilde{\delta}_\omega F(j_{\text{rel}}) = i(2\gamma)^2 F(n_{\text{rel}}) \tag{26}$$

Hence $F(n_{\text{rel}})$ and $F(j_{\text{rel}})$ are eigenvectors of $\tilde{\delta}^2_\omega$:

$$\tilde{\delta}^2_\omega F(n_{\text{rel}}) = (2\gamma)^2 F(n_{\text{rel}}), \qquad \tilde{\delta}^2_\omega F(j_{\text{rel}}) = (2\gamma)^2 F(j_{\text{rel}})$$

yielding the macro-dynamics $\tilde{\alpha}_t$ on the fluctuation operators:

$$\tilde{\alpha}_t F(n_{\text{rel}}) = e^{it\tilde{\delta}_\omega} F(n_{\text{rel}}) = F(n_{\text{rel}}) \cos(2\gamma t) + \frac{1}{2\gamma} F(j_{\text{rel}}) \sin(2\gamma t)$$

$$\tilde{\alpha}_t F(j_{\text{rel}}) = e^{it\tilde{\delta}_\omega} F(j_{\text{rel}}) = -(2\gamma)\, F(n_{\text{rel}}) \sin(2\gamma t) + F(j_{\text{rel}}) \cos(2\gamma t)$$

Proof. This follows immediately from the relations

$$[H^\omega_L, F_L(n_{\text{rel}})] = -iF_L(j_{\text{rel}})$$

$$[H^\omega_L, F_L(j_{\text{rel}})] = i(2\gamma)^2 F_L(n_{\text{rel}}) \quad \blacksquare$$

Remark that we proved that the pair of variables $(F(n_{\text{rel}}), F(j_{\text{rel}}))$ is dynamically independent from the other variables of the system. The pair behaves dynamically as a pair of quantum oscillator variables with a frequency equal to 2γ.

Proposition 4 (Virial theorem). The mean square fluctuation of the relative number operator is proportional to the mean square fluctuation of the relative current operator, in particular:

$$(2\gamma)^2\, \tilde{\omega}(F(n_{\text{rel}})^2) = \tilde{\omega}(F(j_{\text{rel}})^2)$$

Proof. This follows from the time invariance of $\tilde{\omega}$, i.e., $\tilde{\omega} \circ \tilde{\delta}_\omega = 0$:

$$0 = \tilde{\omega}(\tilde{\delta}_\omega[F(n_{\text{rel}})\, F(j_{\text{rel}})]) = \tilde{\omega}(\tilde{\delta}_\omega[F(n_{\text{rel}})]\, F(j_{\text{rel}})) + \tilde{\omega}(F(n_{\text{rel}})\, \tilde{\delta}_\omega[F(j_{\text{rel}})])$$

and the equations of motion (25) and (26). $\blacksquare$

Proposition 5. The mean square fluctuation of the relative number operator is given by

$$\tilde{\omega}(F(n_{\text{rel}})^2) = \frac{c_{\text{rel}}}{4\gamma} \coth \beta\gamma$$

Proof. We compute this quantity using the correlation inequalities (10), rewritten in the form

$$\frac{-\beta\omega(X\delta_\omega(X^*))}{\omega(XX^*)} \leqslant \ln\frac{\omega(X^*X)}{\omega(XX^*)} \leqslant \frac{\beta\omega(X^*\delta_\omega(X))}{\omega(X^*X)} \tag{27}$$

We take for X the operator $A_L = F_L(n_{\text{rel}}) + i(2\gamma)^{-1} F_L(j_{\text{rel}})$ and then let $L \to \infty$, and use Proposition 2.

One gets

$$\lim_{L\to\infty} \omega(A_L A_L^*) = \tilde{\omega}(F(n_{\text{rel}})^2) = (2\gamma)^{-2}\,\tilde{\omega}(F(j_{\text{rel}})^2) + (2\gamma)^{-1}\,c_{\text{rel}}$$

and by the virial theorem (Proposition 4):

$$\lim_{L\to\infty} \omega(A_L A_L^*) = 2\tilde{\omega}(F(n_{\text{rel}})^2) + (2\gamma)^{-1}\,c_{\text{rel}}$$

Analogously

$$\lim_{L\to\infty} \omega(A_L^* A_L) = 2\tilde{\omega}(F(n_{\text{rel}})^2) - (2\gamma)^{-1}\,c_{\text{rel}}$$

On the other hand,

$$\delta_\omega(A_L) = -i(2\gamma)^2\,F_L(j_{\text{rel}}) - (2\gamma)\,F_L(n_{\text{rel}}) = -(2\gamma)\,A_L$$

and hence

$$\lim_{L\to\infty} \omega(A_L^*\delta_\omega(A_L)) = -4\gamma\tilde{\omega}(F(n_{\text{rel}})^2) + c_{\text{rel}}$$

$$\lim_{L\to\infty} \omega(A_L\delta_\omega(A_L^*)) = 4\gamma\tilde{\omega}(F(n_{\text{rel}})^2) + c_{\text{rel}}$$

After substitution in (27) one gets

$$\ln\frac{2\tilde{\omega}(F(n_{\text{rel}})^2) - (2\gamma)^{-1}\,c_{\text{rel}}}{2\tilde{\omega}(F(n_{\text{rel}})^2) + (2\gamma)^{-1}\,c_{\text{rel}}} = -2\beta\gamma$$

or alternatively

$$\tilde{\omega}(F(n_{\text{rel}})^2) = \frac{c_{\text{rel}}}{4\gamma}\coth\beta\gamma \quad \blacksquare$$

This finishes the complete study of the static and dynamic properties of the canonical pair $(F(n_{\text{rel}}), F(j_{\text{rel}}))$ of the relative density and current fluctuations. We proved rigorously that for all temperatures below the condensation temperature and with non-zero condensate, this pair behaves like a pair of quantum harmonic oscillator variables, describing oscillations of the fluid from type 1 into type 2 and vice versa, yielding the typical interference pattern. The plasmon frequency is given by 2γ. All this is physically clear.

Our next and final problem is to find out what the position of the phase is in all this. We turn our attention now to look for a relation between the relative current fluctuation operator $F(j_{\text{rel}})$ and the relative phase fluctuation operator, which we define in an analogous form as the total phase fluctuation operator, as follows:

$$F_L(\phi_{\text{rel}}) = \frac{i}{V^{1/2}} \int_\Lambda dx(b_+^*(x) - b_+(x)) = i(b_{+,0}^* - b_{+,0})$$

Denote its central limit by $F(\phi_{\text{rel}})$.

First, observe that one can distinguish two terms in the relative current fluctuation (24), namely the $k=0$ part and the rest:

$$F_L(j_{\text{rel}}) = \frac{2i\gamma}{V^{1/2}} (b_{+,0}^* b_{-,0} - b_{-,0}^* b_{+,0}) + \frac{2i\gamma}{V^{1/2}} \sum_{k \neq 0} b_{+,k}^* b_{-,k} - b_{-,k}^* b_{+,k}$$

Denote the first term by

$$F_L(j_{\text{rel}}^0) = \frac{2i\gamma}{V^{1/2}} (b_{+,0}^* b_{-,0} - b_{-,0}^* b_{+,0})$$

and its central limit by $F(j_{\text{rel}}^0)$. Also denote by ω^g the ground state, obtained as the zero-temperature limit of the equilibrium state ω:

$$\omega^g(\cdot) = \lim_{\beta \to \infty} \omega(\cdot)$$

and $\tilde{\omega}^g$ the corresponding ground state for the limiting fluctuation operators observables:

$$\tilde{\omega}^g(\cdot) = \lim_{\beta \to \infty} \tilde{\omega}(\cdot)$$

Proposition 6. We have the following relationships between the limiting fluctuation operators:

(i) $\forall \beta > 0$, $\beta = \infty$ included,

$$F(j_{\text{rel}}^0) = 2\gamma \sqrt{\rho_0}\, F(\phi_{\text{rel}})$$

(ii) for $\beta = \infty$,

$$F(j_{\text{rel}}) = F(j^0_{\text{rel}}) = 2\gamma\sqrt{\rho_0}\, F(\phi_{\text{rel}})$$

Proof. As shown in refs. 8 and 9, two fluctuation operators $F(A)$, $F(B)$ are equal in the algebra of fluctuation operators whenever

$$\tilde{\omega}(F(A-B)^2) = 0 \tag{28}$$

i.e., whenever the variance of the difference $A - B$ of the operators vanishes. This is expressing in a mathematical rigorous setting, the phenomenon of coarse graining on the level of fluctuations.

Therefore we calculate

$$\begin{aligned}\tilde{\omega}([F(j^0_{\text{rel}}) - 2\gamma\sqrt{\rho_0}\, F(\phi_{\text{rel}})]^2) = \tilde{\omega}(F(j^0_{\text{rel}})^2) + 4\gamma^2\rho_0\tilde{\omega}(F(\phi_{\text{rel}})^2) \\ - 2\gamma\sqrt{\rho_0}\,\tilde{\omega}(F(j^0_{\text{rel}})\, F(\phi_{\text{rel}})) \\ - 2\gamma\sqrt{\rho_0}\,\tilde{\omega}(F(\phi_{\text{rel}})\, F(j^0_{\text{rel}}))\end{aligned}$$

One finds, using the explicit knowledge of the state ω (Section 3):

$$\tilde{\omega}(F(j^0_{\text{rel}})^2) = 4\gamma^2\rho_0\tilde{\omega}(F(\phi_{\text{rel}})^2) = 4\gamma^2\rho_0 \coth \beta\gamma$$

$$\tilde{\omega}(F(j^0_{\text{rel}})\, F(\phi_{\text{rel}})) = \tilde{\omega}(F(\phi_{\text{rel}})\, F(j^0_{\text{rel}})) = 2\gamma\sqrt{\rho_0} \coth \beta\gamma$$

hence leading to the following equality, as operators:

$$F(j^0_{\text{rel}}) = 2\gamma\sqrt{\rho_0}\, F(\phi_{\text{rel}})$$

From Proposition 4 and 5, it follows that

$$\tilde{\omega}(F(j_{\text{rel}})^2) = c_{\text{rel}}\gamma \coth \beta\gamma$$

and from Proposition 2,

$$\lim_{\beta\to\infty} c_{\text{rel}} = 4\gamma\rho_0$$

Therefore

$$\tilde{\omega}^g(F(j_{\text{rel}})^2) = 4\gamma^2\rho_0 = \tilde{\omega}^g(F(j^0_{\text{rel}})^2)$$

This implies necessarily

$$\tilde{\omega}^g([F(j_{\text{rel}}) - F(j^0_{\text{rel}})]^2) = 0$$

and again the equality of the operators

$$F(j_{\mathrm{rel}}) = F(j^0_{\mathrm{rel}})$$

in the ground state, as a result of coarse graining. ∎

The physical interpretation of this proposition is the following. For non-zero temperatures, the relative current consists of two terms. One of them is j^0_{rel}, which has a non-trivial contribution to the fluctuation of the relative current only if $\rho_0 > 0$, i.e., whenever the gauge symmetry is spontaneously broken. The other term contains no more reference to the zero mode, in other words to the condensate. Therefore it is clear that the fluctuation operator $F(j^0_{\mathrm{rel}})$ contains all the information of the fluctuations of what one could call the *condensate current*, or the current between the condensates interacting through the Josephson junction. The important equality

$$F(j^0_{\mathrm{rel}}) = 2\gamma \sqrt{\rho_0}\, F(\phi_{\mathrm{rel}})$$

is nothing but a rigorous translation, on the level of the fluctuations, of the popular statement: "the (superfluid, condensate) current is the gradient of the phase." The second statement of the proposition shows that the quantum effects on the level of the fluctuations, originating from the spontaneous symmetry breaking, are only present in the ground state. This of course is popular wisdom, already experienced in many models,[10, 14, 22] but expressed here in a mathematically rigorous fashion for our model.

Finally, it may come as a surprise that our results show no dependence on φ, the expectation value for the phase difference between the condensates (see Eq. (15)). This however is a simple consequence of the description of the system in its mathematically simplest form, using the operators $b^{\#}_{\pm,k}$ (8), which yield a φ-independent description of the system. Indeed, from a mathematical point of view, the system can not be expected to behave different for different φ, since e.g., the eigenvalues of the Hamiltonian $E^{\pm}_k$ are φ-independent.

If one is interested in the physics following from a non-zero φ, i.e., if one wants to derive typical Josephson fluctuation currents proportional to $\sin\varphi$, one needs to work with the bare operators $a^{\#}_{(1,2),k}$. In particular, consider the relative current fluctuation operator defined by

$$F_L(j^{\varphi}_{\mathrm{rel}}) = \frac{2\gamma}{V^{1/2}} \sum_k a^*_{1,k} a_{2,k} + a^*_{2,k} a_{1,k} - \omega(a^*_{1,k} a_{2,k} + a^*_{2,k} a_{1,k})$$

in stead of (22).

It can easily be calculated that this operator satisfies

$$\lim_{L\to\infty} [F_L(j_{\text{rel}}), F_L(j^{\varphi}_{\text{rel}})] = 0$$

$$\lim_{L\to\infty} [F_L(n_{\text{rel}}), F_L(j^{\varphi}_{\text{rel}})] = ic_{\text{rel}} \sin\varphi$$

$$\delta^{\omega}(F_L(j^{\varphi}_{\text{rel}})) = i \sin\varphi F_L(n_{\text{rel}})$$

Together with the results above, this establishes that the limiting fluctuation operator $F(j^{\varphi}_{\text{rel}})$ is given by

$$F(j^{\varphi}_{\text{rel}}) = F(j_{\text{rel}}) \sin\varphi \tag{29}$$

where (29) is to be understood in terms of the equivalence between fluctuation operators (28), i.e., (29) follows from

$$\lim_{L\to\infty} \omega((F_L(j^{\varphi}_{\text{rel}}) - \sin\varphi F_L(j_{\text{rel}}))^2) = 0$$

The variances of $F(j^{\phi}_{\text{rel}})$ and its dynamics, computed from Proposition 3 show the explicit φ-dependence, and its typical "collapse and revival" properties, found ref. 11.

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